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On Cooperative Game Approaches For Optimal Portfolio Selection

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Abstract: Cooperative game theory is concerned with exploring schemes for allocating payoffs among rational participants in coalitions and has produced several solution designs due to the different emphasis on criteria such as stability and fairness, but this theory has not been widely applied in the field of portfolio selection. In this paper, we explore further applications of the solution concepts of cooperative games based on the model of optimal portfolio selection developed in previous studies, which is modelled in a static form of a non-cooperative zero-sum game between investors and the market and a cooperative game between investors. We propose a risk modified Shapley value based on the tradeoff between return and risk in the financial market based on the Shapley value, and the performance of this solution shows an evident improvement. We also introduce some other solution concepts of cooperative games and give an approach to construct a nucleolus-based portfolio using Maschler's scheme to compute the nucleolus, and the results demonstrate that the allocation schemes based on the cooperative game theory perform well.

Keywords: Optimal Portfolio Selection, Cooperative Game Theory, Risk Modified Shapley Value, Nucleolus, Stock Market

1. Introduction

In financial markets, investment decisions are made by selecting a range of financial instruments. It is generally believed that it is less risky to diversify investments than to own only one type of asset. When investing in diversified assets, due to the existence of risk and uncertainty, investors always seek to build an optimal portfolio to determine the selection of asset classes, the individual securities within each class, and their corresponding weights. For this purpose, numerous models have been optimized from different perspectives, which often lead to different results due to the different criteria used to measure risk. Modern portfolio theory was greatly advanced by the presentation of Markowitz's portfolio selection theory [1], which provides a mathematical framework for constructing a portfolio from the perspective of mean-variance analysis. To achieve the best tradeoff between return and risk, the optimal portfolio is determined by maximizing return at a given level of risk or minimizing risk at a given level of return. In addition, the variance of asset prices is used to characterize risk and the utility function is introduced to measure how happy or satisfied an economic agent is.

Game theory is the study of mathematical models of strategic choices and interactions between rational decision-makers, which has a variety of applications in the fields of economics and finance.

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Depending on whether there is a binding agreement between interacting parties, game theory can be broadly divided into cooperative games and non-cooperative games. In non-cooperative games, the focus is on the strategic choices of competing antagonists to improve the participants' own utility (or reduce their own cost). The construction of Nash equilibrium theory has contributed significantly to the development of this field, according to which the optimal outcome of a game is one in which no player has an incentive to deviate from his or her chosen strategy after taking the opponents' choices into account. In contrast, cooperative games seek win-win outcomes for participants, i.e., sharing of resources and complementarity of players' strengths, and are primarily concerned with the issue of achieving optimal returns within the coalition and constructing appropriate allocation schemes among coalition members. And to evaluate how good an allocation is, there are numerous criteria such as stability and fairness. By emphasizing different aspects of these criteria, a number of solution concepts have been proposed: the core, the Shapley value, the bargaining set, the nucleolus, the τ value, and so on.

Compared with non-cooperative games, the study of which has long dominated research since John Nash's pioneering work [2], the application of cooperative game theory in the field of financial investment is relatively limited. As cooperative games are also played under risk and uncertainty and aim to maximize the utility function, they share many similarities with optimal portfolio selection. Therefore, it is a natural idea to investigate the application of cooperative game theory to portfolio optimization. Previous research has mainly focused on the application of Shapley value in different market environments and asset class selection. Habip Kocak set participants as investors with different risk appetites (risk-averse player, risk-neutral player, and risk-seeking player), set the counterparty nature to different market environments (balanced market, unbalanced market, and risky market), and finally determined the optimal investment portfolio using the Shapley value [3]. Peyman Tataei, et al. also built the zero-sum model (between investors and the market) and examined that the cooperative game portfolio significantly outperformed the market over 12 years (2006 - 2017) according to the Sharpe and Treynor indices [4]. Ibrahim, et al. studied sectoral portfolio selection before and after the general election in Malaysia and provided a model considering the different stock market sectors as participants versus the market over two time periods [5].

Note that the model structure of cooperative game theory applied to portfolio optimization introduced above is similar and all of them construct allocation schemes based on the Shapley value. However, when the Shapley value is applied to specific research areas, there are still some shortcomings that could be improved. In particular, it does not fully account for the risks borne by different participants. Dai and Xue studied the Shapley value for profit sharing among partners in the case of dynamic coalitions and proposed a risk-based method to improve algorithm performance [6]. Fang, et al. developed and implemented an improved Shapley value approach to achieve the optimal profit sharing for multiple distributed energy resources coexisting in a combined heat and power-virtual power plant problem [7]. Xie, et al. modelled cyber threat intelligence information sharing as a cooperative game problem and proposed a reward mechanism based on Shapley value to make the allocation fair and reasonable [8]. Although the above work has improved the Shapley value for certain applications, the underlying idea of the modification is the same, namely, to construct a vector of risk coefficients within the grand coalition to characterize the risk borne by each individual, differing from the weights that equally shared by the number of participants, and then to increase or decrease the product of this value and the grand coalition utility on top of the original Shapley value for each participant to achieve. It is important to note that the modification needs to be made ensuring that the allocation scheme is included in the imputation set (checking the individual rationality and efficiency of the participants) so that the cooperative game can proceed. We have therefore added this step for the correction compared to the previous design. Regarding the application of portfolio investment, higher returns are generally accompanied by higher risks, but this does not mean that

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high-risk assets must dominate low-risk counterparts in terms of the return. In fact, a range of research focused on the low-risk anomaly, a puzzling phenomenon that assets with lower risks tend to earn higher returns, which is contradictory to the established capital market theory [9][10][11]. Thus, it may not be appropriate to determine the proportion of grand coalition utility to be added (or subtracted) to each investor based solely on the level of risk he or she takes. Instead, to modify the Shapley value considering both return and risk, we use the Sharpe ratio to characterize the risk coefficients vector, i.e. we prefer to assign more grand coalition utility amount to participants with a high Sharpe ratio.

Apart from proposing a risk modified Shapley value for portfolio selection, we also explore the performance of using the nucleolus solution concept to construct the allocation scheme, based on an introduction to the core, the least core, and the Maschler's scheme for computing the nucleolus. Our model results suggest that, in addition to the Shapley value-based construction methods, other cooperative game solution concepts such as the nucleolus could also be effective potential construction ideas.

Following previous studies [3][4][5], the optimal selection problem is characterized by a static model of a non-cooperative zero-sum game between investors and the market to determine the utility values of all coalitions and a cooperative game between investors to decide the payoff allocation among participants. We divide the three game participants according to the market capitalization of the U.S. stock market and use four individual stocks under the corresponding index respectively as available strategies (player A: large-cap companies from the S&P 100 Index, player B: mid-cap companies from the S&P MidCap 400 Index, player C: small-cap companies from the S&P SmallCap 600 Index), while the market with daily stock prices is divided into two periods before and after the outbreak of the Russia-Ukraine conflict. And lastly, the performance of different portfolio schemes is measured using the Sharpe ratio and the Sortino ratio.

2. Cooperative Game Theory

Transferable utility (TU)[1] cooperative game in a characteristic form is denoted by a pair (N, v), where $N = \{1,2,\cdots,n\}$ is a finite set of players in the grand coalition, and v is the characteristic function of the game. The characteristic function with transferable utility is a real-value mapping $v: 2^N \to R$ with $v(\emptyset) = 0$. Any subset S of N is called a coalition, and $v(S)$ represents the maximum utility that the coalition $S \subseteq N$ can achieve, regardless of the actions that players outside the coalition may take. The amount of payoff that player $i \in N$ can receive from the grand coalition payoff $v(N)$ is denoted as x_i, and $x = (x_1, x_2, \cdots, x_n)$ gives an allocation scheme of the game v, where $n = |N|$ is the cardinality of N. It is clear that the vector $x \in R^n$ must satisfy two conditions to ensure that a cooperative game can be formed:

(i) $\sum_{i=1}^{n} x_i = v(N)$ (efficiency condition)

(ii) $x_i \geq v(i),\ \forall i \in N$ (individual rational condition)

We refer to the set formed by all the vectors $x \in R^n$ satisfying the above two conditions as the imputation set $I(v)$. And we focus on the TU cooperative games with superadditivity, which means that

$$v(S) + v(T) \leq v(S \cup T),\ \forall S, T \subseteq N,\ S \cap T = \emptyset. \quad (1)$$

[1] *The utility of a game is transferable if one player can transfer part of the utility to another players without any loss, and such a type of game is called a transferable utility (TU) game. In other words, one can construct an appropriate rule to allocate the obtained coalition utility among the players.*

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The general idea of superadditivity is that if two disjoint subcoalitions S and T cooperate, the payoff of the coalition $S \cup T$ is larger than the sum of the payoffs of these two subcoalitions, i.e. coalitions will yield greater payoffs when they choose to cooperate. If a cooperative game is not superadditive, then coalitions will not tend to cooperate with each other, so the cooperative game problems we study are generally superadditive.

2.1. Risk Modified Shapley Value

The Shapley value is designed by the principle of fair allocation, which means that each player receives a proportional payoff corresponding to the average marginal value of his or her contribution to the coalition. Suppose that a coalition S containing player i can achieve a maximum payoff $v(S)$, while the coalition formed by the players in S excluding i can achieve $v(S\backslash\{i\})$. Therefore, the marginal contribution of player i made to the coalition S is

$$m_i^S = v(S) - v(S\backslash\{i\}). \tag{2}$$

By taking the average of m_i^S over all the different possible permutations in which coalitions could be formed, we can obtain the Shapley value for the player i of game v:

$$\phi_i(v) = \sum_{i \in S \subseteq N} \frac{(s-1)!\,(n-s)!}{n!} \cdot m_i^S, \tag{3}$$

where s and n are the cardinality of S and N respectively. Or alternatively, it can be written as

$$\phi_i(v) = \frac{1}{n} \sum_{i \in S \subseteq N} \binom{n-1}{s-1}^{-1} \cdot m_i^S. \tag{4}$$

An intuitive interpretation of equation (4) is:

$$\phi_i(v) = \frac{1}{\text{number of players}} \sum_{\text{coalitions S including i}} \frac{\text{marginal contribution of i for coalition S}}{\text{number of coalitions including i of size s}}. \tag{5}$$

The vector $\phi(v) = \big(\phi_1(v), \phi_2(v), \cdots, \phi_n(v)\big)$ is called the Shapley value of game v.

However, the Shapley value used to allocate profits has its weaknesses, for example, it does not take the risk borne by each participant in the game into account, which is particularly important in the application of the portfolio selection. Risk is omnipresent when making investment choices, so we should manage risk effectively while expecting higher returns. Participants with a high Sharpe ratio tend to perform better in terms of the return-risk tradeoff, so it is reasonable to increase the returns allocated to participants with a higher Sharpe ratio rather than modifying them solely based on risk. Notice that in the Shapley value allocation scheme, the equation (4) shows that the assumed risk borne by each participant is equivalent (that all equals$1/n$), which means that each participant is assumed to bear the same risk without taking the actual different risks into account. Obviously, this is an ideal situation. In fact, different markets, different financial assets, and different stocks carry different risks. So it is of vital necessarity to introduce a risk-sharing coefficient vector by Sharpe ratio to quantify the return under risk and improve the original results by constructing a modified Shapley value based on this risk-sharing vector. Sharpe ratio is a widely used reward-to-volatility criteria to measure the performance of portfolios, defined as:

$$\text{Sh.r.} := \frac{E\big(R_p\big) - R_f}{\sigma_p} = \frac{\mu_p^T x - R_f}{\sigma_p}, \tag{6}$$

where $E(R_p)$ and σ_p are the expected return and standard deviation of the portfolio, R_f is the risk free rate. We assume $R_f = 0\%$ here, and the expected return is estimated by average return of the portfolio x. It is reasonable that one prefers the higher Sharpe ratio portfolio, i.e. one can get more reward per unit of volatility. Assume the risk-sharing coefficient of player $i \in N$ is R_i, where $\sum_{i \in N} R_i = 1$. Firstly we calculate the Sharpe ratio for each participant $Sh.r.(i)$, and then obtain R_i by normalization, i.e.

$$R_i = \frac{Sh.r.(i)}{\sum_{j=1}^{n} Sh.r.(j)}. \quad (7)$$

The difference between R_i and the equally shared risk is denoted as

$$\Delta R_i := R_i - \frac{1}{n}. \quad (8)$$

Obviously, $\sum_{i \in N} \Delta R_i = 0$. Therefore, the risk modified payoff allocated to player $i \in N$ is equal to $v(N) \cdot \Delta R_i$. Follow the previous studies [6][7][8] on different specific areas, the identical basic idea for proposal risk modified Shapley value is:

$$\phi_i'(v) = \phi_i(v) + v(N) \cdot \Delta R_i. \quad (9)$$

But in fact, to determine whether this proposal scheme could be applied, we need to verify the efficiency condition and individual rational condition:

It is easy to check that the modification preserves the efficiency condition: $\sum_{i \in N} \phi_i'(v) = \sum_{i \in N}(\phi_i(v) + v(N) \cdot \Delta R_i) = \sum_{i \in N} \phi_i(v) + v(N) \cdot \sum_{i \in N} \Delta R_i = v(N)$. To guarantee the individual rational condition holds, we need to check $\phi_i(v) + v(N) \cdot \Delta R_i > v(i),\ \forall i \in N$, which is equivalent to

$$\Delta R_i > \frac{v(i) - \phi_i(v)}{v(N)},\ \forall i \in N. \quad (10)$$

Note that the right hand side of the inequality (10) is a negative value, since $\phi_i(v) > v(i)$ holds for every $i \in N$ in the Shapley value. Thus, for those participants who take a higher risk than the equal average $1/n$ (i.e. $\Delta R_i > 0$), it holds directly. This is because these participants would receive a higher payoff after the modification, and they have no reason to reject the proposal. But the participants who take a lower risk with $\Delta R_i < 0$, they need to give up some of their payoff to others.

Actually, they would still tend to form coalitions if the inequality (10) can be satisfied, even though there is a partial reduction in their payoff. However, if the value of ΔR_i is too low, which means that a participant with very low risk would need to cede so much benefit that the individual rationality condition would be broken, and then the coalition would not be formed. Thus, if the inequality (10) is broken, the proposal is meaningless.

Therefore, based on the equation (9), we design a second-step modification. Subject to satisfying efficiency condition, participant i who breaks inequality (10) after the adjustment is assigned $v(i)$ as the allocation amount, which also means that other participants j, who increase their allocation after adjustment or decrease their allocation after adjustment but are still above $v(j)$, share equally the difference between $v(i)$ and the $\phi_i'(v)$ previously proposed by equation (9). This will be iteratively proceed until a solution is generated that satisfies both efficiency and individual rationality.

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Algorithm 1 Risk modified Shapley value

Input: Coalition utilities $v(S)$, $\forall S \subseteq N$

Output: Risk modified Shapley value $\phi'(v)$

Calculate the Sharpe ratio $Sh.r.(i)$ for each player $i \in N$.
Calculate risk-sharing coefficient $R_i = \frac{Sh.r.(i)}{\sum_{j=1}^{n} Sh.r.(j)}$, then obtain risk modification vector ΔR from $\Delta R_i = R_i - \frac{1}{n}$, $\forall i \in N$.
Calculate Shapley value $\phi_i(v) = \sum_{i \in S \subseteq N} \frac{(s-1)!(n-s)!}{n!} \cdot (v(S) - v(S \backslash \{i\}))$, $\forall i \in N$.
Set initial value $\phi'_i(v) = \phi_i(v) + v(N) \cdot \Delta R_i$, $\forall i \in N$.
while $\phi'_1(v) < v(1)$, or $\phi'_2(v) < v(2)$, $\cdots$, or $\phi'_n(v) < v(n)$, or $\sum_{i=1}^{n} \phi'_i(v) \neq v(N)$ **do**
    $num = 0$
    $sum = 0$
    for all $i = 1 : n$ **do**
        if $\phi'_i(v) \leq v(i)$ **then**
            $num = num + 1$
            $sum = sum + |v(i) - \phi'_i(v)|$
        end if
    end for
    for all $i = 1 : n$ **do**
        if $\phi'_i(v) > v(i)$ **then**
            $\phi'_i(v) = \phi'_i(v) - \frac{1}{n - num} \cdot sum$
        else
            $\phi'_i(v) = v(i)$
        end if
    end for
end while
Return $\phi'(v)$

Figure 1. Algorithm 1.

2.2. Nucleolus

In addition to the Shapley value, which distributes payoffs according to marginal contribution, there are a number of other static allocation schemes within the grand coalition. The core, which addresses the fairness and stability of allocations within the coalition, guarantees that neither an individual player nor a subcoalition is inclined to leave the grand coalition in order to achieve a higher payoff. More precisely, the core $C(v)$ of v is defined as:

$$C(v) \coloneqq \{x \in R^n | \sum_{i \in N} x_i = v(N), \quad \sum_{i \in S} x_i \geq v(S), \ \forall S \subset N\}. \tag{11}$$

Note that the core of a cooperative game can be either an area (or a point), or empty. To overcome these two main difficulties, nonexistence and nonuniqueness, one can strengthen or relax the definitional inequalities of the core. It is a natural idea to consider the least core [12][13]. The excess of coalition S with respect to the allocation scheme x is given by

$$e(S, x) \coloneqq v(S) - \sum_{i \in S} x_i \, . \tag{12}$$

The excess is used to measure the unhappiness of the coalition S under current allocation scheme x. The larger this value is, the unhappier the coalition S will be with x. From the definition of the core, x is in the core if and only if all the excesses are nonpositive. By adding a uniform slack ϵ to the inequalities of each subcoalition in the core, it gives the definition of the ϵ-core:

$$C_\epsilon(v) \coloneqq \{x \in R^n | \sum_{i \in N} x_i = v(N), \quad \sum_{i \in S} x_i \geq v(S) - \epsilon, \ \forall S \subset N\}. \tag{13}$$

The least core is the set of all possible schemes x satisfying (13) for the minimum ϵ. Hence, we can obtain the least core by minimizing the maximum excess of all subcoalitions:

$$C_{\min \epsilon}(v) \coloneqq \min_{x \in C_\epsilon(v)} \epsilon = \min_{x \in C_\epsilon(v)} \max_{S \subset N} e(S, x) \ . \tag{14}$$

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Or alternatively, the least core can be viewed as solving a linear program:

$$\begin{aligned} \text{minimize} \quad & \epsilon \\ \text{subject to} \quad & e(S,x) \leq \epsilon, \forall S \subset N \\ & \sum_{i \in N} x_i = v(N) \end{aligned} \tag{15}$$

The nucleolus of a TU cooperative game (N, v) is defined as:

$$Nu(v) := \{ x \in I(v) \mid \theta(x) \leq_L \theta(y), \forall y \in I(v) \}, \tag{16}$$

where the vector $\theta(x)$ is constructed by sorting the excess of all subsets of the grand coalition N in decreasing order, i.e. $\theta_i(x) \geq \theta_j(x)$ for $1 \leq i \leq j \leq 2^n$. We use $\alpha <_L \beta$ to denote that α is lexicographically smaller than β, and $\alpha \leq_L \beta$ indicates either $\alpha <_L \beta$ or $\alpha = \beta$. The nucleolus of a TU cooperative game always consists of a point that in the core whenever the core is nonempty [14]. In fact, the computation of the nucleolus can be obtained by solving a series of linear programs $(O_i)_{i=1,2,\cdots}$ recursively. Let $e_1(x), e_2(x), \cdots$ sorted in decreasing order be the excesses of all $2^n - 2$ coalitions (except $\emptyset$ and N) with respect to x. The linear program (15) for calculating the least core is equivalent to minimizing $e_1(x)$. Use ϵ_i and $P_i(\epsilon_i)$ to denote the optimal objective value and the set of coalitions for which the constraints are realized as equalities in the optimal solution to the optimization program (O_i). Obviously, $\epsilon_1 > \epsilon_2 > \cdots$, $P_1(\epsilon_1) \subset P_2(\epsilon_2) \subset \cdots$. Then recursively minimize the second largest excess $e_2(x)$, the third largest excess $e_3(x)$, and so on, until there exists L such that $P_L(\epsilon_L)$ contains all the nonempty subsets of N, i.e. $P_L(\epsilon_L) = 2^N \backslash \emptyset$. This recursive algorithm is called the Maschler's scheme [15][16], which eventually leads to a single vector, the nucleolus.

Algorithm 2 Nucleolus: Maschler's scheme

Input: Coalition utilities $v(S)$, $\forall S \subseteq N$

Output: Nucleolus x

1:

$$(O_1) \begin{cases} \text{minimize} & \epsilon \\ \text{subject to} & e(S,x) \leq \epsilon, \ \forall S \subset N \\ & \sum_{i \in N} x_i = v(N) \end{cases}$$

2: **for all** $i = 2 : L$ **do**

$$(O_i) \begin{cases} \text{minimize} & \epsilon \\ \text{subject to} & e(S,x) = \epsilon_1, \ \forall S \in P_1(\epsilon_1) \\ & \vdots \\ & e(S,x) = \epsilon_{i-1}, \ \forall S \in P_{i-1}(\epsilon_{i-1}) \backslash P_{i-2}(\epsilon_{i-2}) \\ & e(S,x) \leq \epsilon, \ \forall S \subset 2^N \backslash P_{i-1}(\epsilon_{i-1}) \\ & \sum_{i \in N} x_i = v(N) \end{cases}$$

3: **end for**

4: Return x

Figure 2. Algorithm 2.

Note that Maschler's scheme might be efficient for games with a small number of participants. However, for a large number of participants, this method is limited by the large computational complexity of the exponential scale, which prevents satisfactory solutions from being obtained at a high computational speed. When this situation occurs, we can consider some other algorithms instead. For example, Durga Prasad proposed an algorithm to obtain the nucleolus in solving at most

$(n-1)$ linear programs for n-player games, where some constraints are dropped at each iteration without affecting the unique final solution [17].

3. Data and Methodology

Standard&Poor's (S&P) is a company with a global reputation as a creator of financial market indices and is widely used as a data source for investment benchmarking. The S&P 100 Index is a subset of the famous S&P 500 Index, designed to measure the performance of large-cap U.S. companies, and consists of 100 major blue chip companies across a range of industry sectors. In addition, the S&P MidCap 400 Index provides investors with a benchmark of 400 mid-cap companies, reflecting the unique risk and return characteristics of this market segment. And the S&P SmallCap 600 Index is designed to evaluate the performance of the small-cap segment of the U.S. equity market, with the goal of tracking the financial viability and liquidity of companies that meet certain inclusion criteria. All data is sourced from Yahoo Finance.

This research is modelled as a zero-sum2 non-cooperative game between investors and the market and a cooperative game among investors in a static3 form. For the market, its movement is influenced by many factors, e.g., the economy, inflation, and politics, etc. In particular, [5] explores the systematic impact of political events (general elections) on financial markets and uses pre- and post-election phases to characterize market strategies. In the era of globalization of economy and trade, a geopolitical crisis in one region often affects not just the countries in that region, but the whole world. Since the outbreak of the Russia-Ukraine conflict, there have been significant and long-lasting impacts on the global economy and financial markets on all fronts. This not only has a direct negative influence on energy supply and inflationary pressures, but also leads to a number of potential indirect risks such as slower consumer spending, supply chain distortions, credit and asset write-downs, and tighter monetary policies, etc. [18][19]. Therefore, in this paper we will try to distinguish market strategies using two phases before and after the start of the Russia-Ukraine war. Ever since Russia's invasion of Ukraine on 24 February 2022, a historic series of changes across global financial markets have occurred. From this perspective, We choose this date as the segmentation node to reflect the two dynamics within the market.

Period 1 (P1): 1 September 2021 -- 23 February 2022, 120 trading days in total.

Period 2 (P2): 24 February 2022 -- 1 September 2022, 131 trading days in total.

For the investor, it is assumed that the participants are differentiated by market capitalization of the companies, using players A, B, C to represent the investment in large-cap, mid-cap and small-cap U.S. companies respectively. Note that companies with larger market capitalization tend to be more established and therefore considered to be less risky, but at the same time it may imply a smaller potential return. On the other hand, the younger small-cap companies tend to serve new industries and niche market, which are generally seen as the riskiest of the three types. Accordingly, the strategies (stocks) of each player are given by the several top constituents by index weight of the corresponding S&P Index. i.e.

A: Large-cap companies, chosen from the constituents of S&P 100 Index.

B: Mid-cap companies, chosen from the constituents of S&P MidCap 400 Index.

C: Small-cap companies, chosen from the constituents of S&P SmallCap 600 Index.

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[2] *Zero-sum games are non-cooperative games in which the parties involved are in a strictly competitive situation, where a gain for one party implies a loss for the other, and where the sum of gains and losses for all parties always adds up to zero.*

[3] *Static games are games in which participants choose simultaneously or not simultaneously but the later actor does not know what decision the first actor has taken. In contrast, in a dynamic game, participants act sequentially and the later actor is able to observe the decision of the first actor.*

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Table 1. Players and strategies.

Player	Strategy	Symbol	Name	Sector
A	A1	AAPL	Apple Inc.	Information Technology
	A2	MSFT	Microsoft Corp.	Information Technology
	A3	AMZN	Amazon.com Inc.	Consumer Discretionary
	A4	TSLA	Tesla Inc.	Consumer Discretionary
B	B1	TRGP	Targa Resources Corp.	Energy
	B2	CSL	Carlisle Cos.	Industrials
	B3	STLD	Steel Dynamics Inc.	Materials
	B4	EQT	EQT Corp.	Energy
C	C1	ADC	Agree Realty Corp.	Real Estate
	C2	EXLS	ExlService Holdings Inc.	Information Technology
	C3	LNTH	Lantheus Holdings Inc.	Health Care
	C4	SM	SM Energy Corp.	Energy

The daily adjusted closing price of each stock is used to calculate the return rate. Assume $P_{i,t}$ is the daily adjusted closing price of stock i at time t, then the rate of return of the stock i at time t is

$$r_{i,t} = \frac{P_{i,t} - P_{i,t-1}}{P_{i,t-1}} \tag{17}$$

for $i = 1, \cdots, 12$ and $t = 1, \cdots, T$, where T is the number of returns obtained in period 1 or period 2. We use the geometric average to estimate the mean return rate of each stock at these two time periods respectively, i.e.

$$\mu_i = \left[\prod_{t=1}^{T}\left(1 + r_{i,t}\right)\right]^{\frac{1}{T}} - 1. \tag{18}$$

By explaining the behavior of investors in a cooperative game perspective, we try to give optimal portfolio selection schemes not necessarily the best one but the best available choices according to the cooperative game solution concepts within the Nash Equilibrium range. Investors and the market are in conflict (zero-sum game) and the payoff gained by the investors is treated as a loss of the same amount to the market. Denote the payoff matrix as $Q_{m\times 2}$, where the row number m indicates the strategies that can be chosen in coalition S for the investors while there are only two choices (P1 or P2) for the market. For the investors, they always tend to construct a portfolio vector $x_{m\times 1}$ among the optional strategies they can choose to maximize the possible minimum payoff throughout two periods. Assume no short selling. The optimal value for each coalition is the following optimization:

$$(\text{Investor})\begin{cases}\text{maximize} & v\\ \text{subject to} & Q^T \cdot x \geq v \cdot 1_{2\times 1}\\ & 1_{1\times m} \cdot x = 1\\ & x \geq 0\end{cases} \tag{19}$$

On the contrary, since investors' gain is actually the market's loss, the market player chooses the probability vector $y_{2\times 1}$ to obtain the optimal solution by the following linear program:

$$(\text{Market})\begin{cases}\text{minimize} & \text{u}\\ \text{subject to} & \text{Q}\cdot\text{y}\le\text{u}\cdot 1_{\text{m}\times 1}\\ & 1_{1\times 2}\cdot\text{y}=1\\ & \text{y}\ge 0\end{cases} \tag{20}$$

Denote the optimal values for linear program (19) and (20) as v^* and u^* respectively. Actually one can show that $\text{v}^*=\text{u}^*$, and this balanced state achieves the Nash Equilibrium. Or alternatively, we can combine (19) and (20) into an optimization problem:

$$\begin{array}{ll}\text{maximize} & \text{v}-\text{u}\\ \text{subjectto} & \text{Q}^{\text{T}}\cdot\text{x}\ge\text{v}\cdot 1_{2\times 1}\\ & \text{Q}\cdot\text{y}\le\text{u}\cdot 1_{\text{m}\times 1}\\ & 1_{1\times\text{m}}\cdot\text{x}=1\\ & 1_{1\times 2}\cdot\text{y}=1\\ & \text{x}\ge 0\\ & \text{y}\ge 0\end{array} \tag{21}$$

Assume the optimal solution for linear programs (21) is x^*,y^*. Then we can also obtain the coalition utility $\text{v}(\text{S})$ from the sum of the products of each element of the payoff matrix and their corresponding probabilities4, i.e.

$$\text{v}(\text{S})=\text{v}^*=\text{u}^*=\text{Q}\odot(\text{x}^*\cdot\text{y}^{*\text{T}}). \tag{22}$$

Using the geometric mean return (18) of each stock at both two time periods, we obtain the payoff matrix for three individual players A, B, C:

Table 2. Payoff matrix structured for $\{A\}$ **Table 3.** Payoff matrix structured for $\{B\}$ **Table. 4** Payoff matrix structured for $\{C\}$

Strategy	P1	P2	Strategy	P1	P2	Strategy	P1	P2
A1	0.00065	-0.00024	B1	0.00283	0.00080	C1	-0.00125	0.00144
A2	-0.00037	-0.00088	B2	0.00093	0.00195	C2	-0.00064	0.00275
A3	-0.00123	-0.00136	B3	-0.00045	0.00195	C3	0.00066	0.00520
A4	0.00095	0.00025	B4	0.00125	0.00596	C4	0.00494	0.00182

By the CVX optimization toolbox, a modeling system for convex optimization problems on MATLAB, we can use the equation (22) to obtain: $\text{v}(\text{A})=0.00025$, $\text{v}(\text{B})=0.00235$, $\text{v}(\text{C})=0.00320$.

Furthermore, for the payoff matrix of coalitions containing two participants, the payoff for an arbitrary strategy at a given period is equal to the sum of that of two individual players. For example, the payoff for $(\text{A1B1},\text{P1})$ is equal to the sum of payoffs of $(\text{A1},\text{P1})$ and $(\text{B1},\text{P1})$. After obtaining the payoff matrix Q in this way, we can use optimization (22) to get the utilities for two participants' coalitions.

[4] *Hadamard product $\odot$: If $A=(a_{ij})$ and $B=(b_{ij})$ are two matrices of the same order, then $(A\odot B)_{ij}=a_{ij}\times b_{ij}$.*

Table 5. Payoff matrix structured for $\{A, B\}$

Strategy	P1	P2
A1B1	0.00348	0.00056
A1B2	0.00158	0.00170
A1B3	0.00020	0.00170
A1B4	0.00190	0.00572
A2B1	0.00246	-0.00008
A2B2	0.00056	0.00106
A2B3	-0.00082	0.00106
A2B4	0.00088	0.00508
A3B1	0.00159	-0.00056
A3B2	-0.00031	0.00058
A3B3	-0.00168	0.00058
A3B4	0.00001	0.00460
A4B1	0.00377	0.00105
A4B2	0.00187	0.00219
A4B3	0.00050	0.00219
A4B4	0.00219	0.00621

Table 6. Payoff matrix structured for $\{A, C\}$

Strategy	P1	P2
A1C1	-0.00060	0.00119
A1C2	0.00001	0.00250
A1C3	0.00131	0.00496
A1C4	0.00559	0.00158
A2C1	-0.00162	0.00055
A2C2	-0.00101	0.00187
A2C3	0.00029	0.00432
A2C4	0.00457	0.00094
A3C1	-0.00249	0.00007
A3C2	-0.00187	0.00139
A3C3	-0.00057	0.00384
A3C4	0.00371	0.00046
A4C1	-0.00031	0.00168
A4C2	0.00031	0.00299
A4C3	0.00161	0.00545
A4C4	0.00589	0.00207

Table 7. Payoff matrix structured for $\{B, C\}$

Strategy	P1	P2
B1C1	0.00157	0.00224
B1C2	0.00219	0.00355
B1C3	0.00349	0.00600
B1C4	0.00777	0.00262
B2C1	-0.00033	0.00338
B2C2	0.00029	0.00469
B2C3	0.00159	0.00715
B2C4	0.00587	0.00377
B3C1	-0.00170	0.00338
B3C2	-0.00109	0.00469
B3C3	0.00021	0.00715
B3C4	0.00449	0.00377
B4C1	-0.00001	0.00740
B4C2	0.00061	0.00871
B4C3	0.00191	0.01116
B4C4	0.00619	0.00778

Hence, we have $v(\{A, B\}) = 0.00313$, $v(\{A, C\}) = 0.00375$, $v(\{B, C\}) = 0.00656$. In the same way, the payoff matrix for the grand coalition is given as below:

Table 8. Payoff matrix structured for $\{A, B, C\}$.

Strategy	P1	P2	Strategy	P1	P2	Strategy	P1	P2
A1B1C1	0.00222	0.00199	A2B2C3	0.00122	0.00626	A3B4C1	-0.00124	0.00603
A1B1C2	0.00284	0.00331	A2B2C4	0.00550	0.00288	A3B4C2	-0.00063	0.00735
A1B1C3	0.00414	0.00576	A2B3C1	-0.00207	0.00250	A3B4C3	0.00067	0.00980
A1B1C4	0.00842	0.00238	A2B3C2	-0.00146	0.00381	A3B4C4	0.00495	0.00642
A1B2C1	0.00032	0.00314	A2B3C3	-0.00015	0.00626	A4B1C1	0.00252	0.00248
A1B2C2	0.00094	0.00445	A2B3C4	0.00412	0.00289	A4B1C2	0.00313	0.00380
A1B2C3	0.00224	0.00690	A2B4C1	-0.00038	0.00651	A4B1C3	0.00443	0.00625
A1B2C4	0.00652	0.00352	A2B4C2	0.00024	0.00783	A4B1C4	0.00871	0.00287
A1B3C1	-0.00105	0.00314	A2B4C3	0.00154	0.01028	A4B2C1	0.00062	0.00363
A1B3C2	-0.00044	0.00445	A2B4C4	0.00582	0.00690	A4B2C2	0.00123	0.00494
A1B3C3	0.00086	0.00690	A3B1C1	0.00034	0.00087	A4B2C3	0.00253	0.00739
A1B3C4	0.00514	0.00353	A3B1C2	0.00095	0.00219	A4B2C4	0.00681	0.00401
A1B4C1	0.00064	0.00715	A3B1C3	0.00226	0.00464	A4B3C1	-0.00075	0.00363
A1B4C2	0.00126	0.00846	A3B1C4	0.00653	0.00126	A4B3C2	-0.00014	0.00494
A1B4C3	0.00256	0.01092	A3B2C1	-0.00156	0.00202	A4B3C3	0.00116	0.00739
A1B4C4	0.00684	0.00754	A3B2C2	-0.00095	0.00333	A4B3C4	0.00544	0.00402
A2B1C1	0.00120	0.00135	A3B2C3	0.00035	0.00578	A4B4C1	0.00094	0.00764
A2B1C2	0.00182	0.00267	A3B2C4	0.00463	0.00240	A4B4C2	0.00155	0.00895
A2B1C3	0.00312	0.00512	A3B3C1	-0.00293	0.00202	A4B4C3	0.00285	0.01141
A2B1C4	0.00740	0.00174	A3B3C2	-0.00232	0.00333	A4B4C4	0.00713	0.00803
A2B2C1	-0.00070	0.00250	A3B3C3	-0.00102	0.00578			
A2B2C2	-0.00008	0.00381	A3B3C4	0.00326	0.00241			

The utility value for the grand coalition is $v(\{A,B,C\}) = 0.00734$. Therefore, the cooperative game (N,v) among three participants can be modelled as:

Table 9. Characteristic function for cooperative game (N,v).

Utility	$v(\emptyset)$	$v(A)$	$v(B)$	$v(C)$	$v(\{AB\})$	$v(\{AC\})$	$v(\{BC\})$	$v(N)$
Value	0	0.00025	0.00235	0.00320	0.00313	0.00375	0.00656	0.00734

According to the equation (4), the Shapley value vector is

$$\phi(v) = (0.00057, 0.00302, 0.00376)^T. \quad (23)$$

After the normalization, we can obtain the proportions that belong to three participants are $P(A) = 0.07698$, $P(B) = 0.41144$, $P(C) = 0.51158$. Assume the probabilities that the strategies A_i, B_j, C_k occur to achieve optimal individual utilities $v(A)$, $v(B)$, $v(C)$ are α_i^*, β_j^*, γ_k^* respectively, where $i = 1,\cdots,4$, $j = 1,\cdots,4$, $k = 1,\cdots,4$. And from the previous results of the optimization (21) for three individual player coalitions, we have

$$\begin{cases} \alpha^* = (0,0,0,1)^T \\ \beta^* = (0.69942,0,0,0.30058)^T \\ \gamma^* = (0,0,0.40713,0.59287)^T \end{cases} \quad (24)$$

Then, we can determine the 12 individual stocks' weights in the entire portfolio by using these vectors times the corresponding proportions with respect to the participants, i.e. $w_i = P(A)\cdot\alpha_i^*$, $w_j = P(B)\cdot\beta_j^*$, $w_k = P(C)\cdot\gamma_k^*$. Therefore, the weights vector of the Shapley value portfolio is

$$w = (0,0,0,0.07698,0.28777,0,0,0.12367,0,0,0.20828,0.30330)^T. \quad (25)$$

Now we perform the risk adjustments to the Shapley value. The risk is measured with variance here, and the square root of the variance gives the standard deviation of the portfolio. The average return and the variance of the portfolio are given as follow:

$$\begin{aligned} \mu_p &= \textstyle\sum_{i=1}^{k} w_i\mu_i \\ \sigma_p^2 &= \textstyle\sum_{i=1}^{k}\sum_{j=1}^{k} w_i w_j \sigma_{ij} \end{aligned} \quad (26)$$

where w_i is the proportion of the portfolio invested in the ith stock, μ_i is the average return of the ith stock, σ_{ij} is the covariance between the ith and jth stocks (σ_{ii} denotes the variance of ith stock), and k is the number of stocks in the portfolio. For example, the average return, standard deviation and Sharpe ratio for the Shapley value portfolio (25) are: $\mu_p = 0.00265$, $\sigma_p = 0.02612$, $Sh.r. = 0.10133$.

To take different participants' risk into consideration, the covariance matrix for each individual player's optimal portfolio is built in the analysis. By equation (26), we can calculate the average return and standard deviation of the optimal portfolio of player A, B, C according to the weights vector α^*, β^*, and γ^* respectively. The results are: $\mu_A = 0.00058$, $\sigma_A = 0.03972$; $\mu_B = 0.00235$, $\sigma_B = 0.02436$; $\mu_C = 0.00320$, $\sigma_C = 0.03248$. The Sharpe ratio vector with respect to the participants is

$$Sh.r. = (0.01462, 0.09649, 0.09839)^T. \quad (27)$$

After the normalization, we obtain the risk-sharing coefficient vector $R = (0.06979, 0.46059, 0.46963)$. Since $\Delta R_i = R_i - 1/n$, where n is the number of participants, we subtract $1/3$ for each element in the vector to get

$$\Delta R = (-0.26355, 0.12726, 0.13629)^T. \tag{28}$$

Through the proposed algorithm 1, the risk modified Shapley value is

$$\phi'(v) = (0.00025, 0.00314, 0.00395)^T. \tag{29}$$

By normalization, we can obtain the modified proportion of each participant $P(A) = 0.03406$, $P(B) = 0.42838$, $P(C) = 0.53756$. Therefore, the weights vector for the risk modified Shapley value portfolio is

$$w = (0,0,0,0.03406,0.29962,0,0,0.12876,0,0,0.21886,0.31870)^T. \tag{30}$$

Moreover, according to Maschler's scheme introduced in algorithm 2, we can calculate the nucleolus of the cooperative game shown in Table 9 as follow:

$$\text{nucl} = (0.00052, 0.00310, 0.00372)^T. \tag{31}$$

After the normalization, we obtain $P(A) = 0.07016$, $P(B) = 0.42268$, $P(C) = 0.50715$. Apply the same calculation as above, the weights vector of the nucleolus portfolio is

$$w = (0,0,0,0.07016,0.29563,0,0,0.12705,0,0,0.20648,0.30067)^T. \tag{32}$$

4. Results and Discussion

The performance of different portfolio schemes is measured by expected rate of return, standard deviation, Sharpe ratio and Sortino ratio. Different from the Sharpe ratio which uses the standard deviation, Sortino ratio performs the lower partial standard deviation to differentiate between adverse and favourable fluctuations. The interpretation for this is that the upside volatility with positive return of the portfolio satisfies the investor's needs and should not be adjusted. It is defined as

$$\text{Sor.r.} := \frac{E(R_p) - \text{MAR}}{\sqrt{\text{SemiVar}_p}} = \frac{\mu_p^T x - \text{MAR}}{\sqrt{E\left[\left(\min\{0, (r_p - \mu_p)^T x\}\right)^2\right]}}, \tag{33}$$

where MAR is the minimum acceptable return, in most cases it is set equal to the risk free rate R_f, and we assume $\text{MAR} = 0\%$ here. Like the Sharpe ratio, investors tend to prefer the portfolio with a higher Sortino ratio.

For comparison, we also give the market portfolio and the naive diversification portfolio to illustrate the effectiveness of the investment schemes based on the cooperative game solution concepts. The market portfolio is to invest in assets in proportion to the market's current valuation of them, in other words, to build a portfolio weighting 12 stocks by their market capitalization as a proportion of the total's, i.e. $w = (0.38435, 0.28210, 0.18429, 0.13716, 0.00215, 0.00225, 0.00204, 0.00242, 0.00091, 0.00088, 0.00075, 0.00070)^T$. Naive diversification of the portfolio is achieved by equally distributing its weights among 12 stocks using $1/12$, i.e. $w = (0.08333, 0.08333, 0.08333, 0.08333, 0.08333, 0.08333, 0.08333, 0.08333, 0.08333, 0.08333, 0.08333, 0.08333)^T$. So far, we have obtained five investment schemes as shown in Table 10: market portfolio, naive diversification portfolio, Shapley value portfolio, nucleolus portfolio, risk modified Shapley value portfolio. And the performance of each scheme is presented in Table 11.

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Table 10. Portfolio weights of different schemes.

Player	Strategy	Symbol	Portfolio weights				
			I	II	III	IV	V
A	A1	AAPL	0.38435	0.08333	0	0	0
	A2	MSFT	0.28210	0.08333	0	0	0
	A3	AMZN	0.18429	0.08333	0	0	0
	A4	TSLA	0.13716	0.08333	0.07698	0.07016	0.03406
B	B1	TRGP	0.00215	0.08333	0.28777	0.29563	0.29962
	B2	CSL	0.00225	0.08333	0	0	0
	B3	STLD	0.00204	0.08333	0	0	0
	B4	EQT	0.00242	0.08333	0.12367	0.12705	0.12876
C	C1	ADC	0.00091	0.08333	0	0	0
	C2	EXLS	0.00088	0.08333	0	0	0
	C3	LNTH	0.00075	0.08333	0.20828	0.20648	0.21886
	C4	SM	0.00070	0.08333	0.30330	0.30067	0.31870

I: Market portfolio, II: Naive diversification portfolio, III: Shapley value portfolio, IV: Nucleolus portfolio, V: Risk modified Shapley value portfolio.

Table 11. Performance of different schemes.

Portfolio	I	II	III	IV	V
Expected return rate	-0.00025	0.00118	0.00265	0.00266	0.00274
Standard deviation	0.02029	0.01747	0.02612	0.02611	0.02660
Sharpe ratio	-0.01213	0.06762	0.10133	0.10170	0.10319
Sortino ratio	-0.01668	0.09241	0.14012	0.14058	0.14274

I: Market portfolio, II: Naive diversification portfolio, III: Shapley value portfolio, IV: Nucleolus portfolio, V: Risk modified Shapley value portfolio.

By comparing the results, we can intuitively observe that the allocation schemes based on the cooperative game solution concepts significantly outperform the cases of the market portfolio and the naive diversification portfolio. For the risk modified Shapley value portfolio, the Sharpe ratio and Sortino ratio of that are 0.10319 and 0.14274, which is an improvement over the results of the original Shapley valued-based portfolio. In addition, the Sharpe ratio and the Sortino ratio of the necleolus portfolio are 0.10170 and 0.14058, showing that the allocation scheme based on the nucleolus also achieves the desired outcome.

However, although the findings demonstrate an apparent improvement of risk modified Shapley value scheme on original Shapley value scheme and a promising application of the nucleolus solution concept under the present model, it is not appropriate to compare different solution concepts and draw general conclusions based solely on the results of the present case. This is because many solution concepts, including the Shapley value and the nucleolus, focus on different aspects of fairness, stability or other criteria, and the relative relationships they perform may vary with different specific situations.

Furthermore, the model used in this paper still has some shortcomings, one of which is the complexity of the computation. Note that this paper is actually modelling three participants with four strategies of each, and that the computational complexity will increase exponentially in scale when increasing the number of participants, so it may take longer to perform large-scale calculations (e.g. including a sufficient number of sectors as participants and using numerous individual stocks under each sector as strategies). On the other hand, the modelling of cooperative game theory applied to portfolio selection problem is limited to static games, while the study of dynamic game processes and their corresponding asset management is still lacking.

5. Conclusions

In this paper, we further explore the application of cooperative game theory to the optimal portfolio selection problem, and list several allocation schemes for portfolio weights based on the cooperative game solution concepts. Under a static model of a zero-sum non-cooperative game between investors and the market and a cooperative game among investors, we divide the three game participants by the market capitalization of the U.S. stock market and use the individual stocks under the corresponding index as the available strategies, while the market is divided into two periods of daily stock prices before and after the outbreak of the Russia-Ukraine conflict. Our main contribution is to propose a risk modified Shapley value based on the Shapley value by redistributing the payoff received within the grand coalition according to the return-risk balance of each participant, and the model results demonstrate the effectiveness of this improvement. And we show other solution concepts for cooperative games can also be used in this model, for example the construction of nucleolus-based portfolio has also produced satisfactory outcome in terms of return and risk.

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